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A fixed point result on an extended neutrosophic rectangular metric space with application

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Abstract

In this paper, we propose the notion of extended neutrosophic rectangular metric space and prove some fixed point results under contraction mapping. Finally, as an application of the obtained results, we prove the existence and uniqueness of the Caputo fractional differential equation.

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1 Introduction

The foundation of fixed point theory consists of the notion of metric spaces and the Banach contraction principle [1]. The spaciousness of metric space is attracting thousands of academics with its axiomatic interpretation see [2–10]. There have been numerous metric space generalisations made recently. This demonstrates the elegance, allure and growth of the idea of metric spaces. The notion of fuzzy sets was proposed by Zadeh [11]. The term "fuzzy" appears to be widely used and frequently occurring in current research on the logical and set-theoretical foundations of mathematics. We believe that the primary reason for this rapid development is simple to understand. The world we live in is full of uncertainty because, for the most part, the data that come from our findings and measurements, the ideas we utilise and the information we gather from the environment are all imprecise and ambiguous. Therefore, any formal description of the real world, or parts of it, is always merely an idealisation and an approximation of the true reality. Fuzzy sets, fuzzy orderings, fuzzy languages and so on allow us to investigate and deal with the previously specified degree of uncertainty in a strictly formal and mathematical manner. The concept of fuzzy sets has succeeded in shifting a lot of mathematical structures within its concept. Schweizer and Sklar [12] defined the notion of continuous t-norms. Kramosil and Michalek [13] introduced the notion of fuzzy metric spaces. They applied the concept of fuzziness, via continuous t-norms, to classical notions of metric and metric spaces and compared the notions thus obtained with those resulting from some other, namely probabilistic, statistical generalisations of metric spaces. Garbiec [14] provided the fuzzy interpretation of Banach contraction principle in fuzzy metric spaces. Ur-Reham et al. [15] proved some $\alpha - \phi$-fuzzy cone contraction results with integral type application. Fuzzy metric spaces only deal with membership functions. An intuitionistic fuzzy metric space that is used to deal with both membership and non-membership functions was established by Park [16]. Konwar [17] presented the concept of an intuitionistic fuzzy b-metric space and proved several fixed point theorems. Kirişci and Simsek [18] introduced the notion of neutrosophic metric spaces that is used to deal with membership, non-membership and naturalness. Simsek and Kirişci [19] proved some amazing fixed point results in the context of neutrosophic metric spaces. Sowndrarajan et al. [20] proved some fixed point results in the setting of neutrosophic metric spaces. Itoh [21] proved an application regarding random differential equations in Banach spaces. Mlaiki [22] coined the concept of controlled metric spaces and proved several fixed point results for contraction mappings. Sezen [23] presented the notion of controlled fuzzy metric spaces and proved various contraction mapping results. Recently, Saleem et al. [24] introduced the concept of fuzzy double controlled metric spaces. For related articles, see [25, 26, 30–33]. In 2022, Uddin et al. [27] proved fixed point theorem on neutrosophic double controlled metric space. In 2022, Gunaseelan et al. [28] proposed neutrosophic rectangular triple controlled metric space and proved fixed point theorems.

In this paper, we introduce the notion of extended neutrosophic rectangular metric space and prove fixed point theorems. The main objectives of this paper are as follows:

- Introduce the notion of extended neutrosophic rectangular metric space;
- Prove several fixed point theorems for contraction mappings;
- Find the existence and uniqueness solution of the fractional differential equation with boundary conditions.

2 Preliminaries

In this section, we provide some definitions that will be helpful for readers to understand the main section.

Definition 1 [16] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if:

1. $\wp * \tau = \tau * \wp$ for all $\wp, \tau \in [0,1]$;
2. $*$ is continuous;
3. $\wp * 1 = \wp$ for all $\wp \in [0,1]$;
4. $(\wp * \tau) * \mu = \wp * (\tau * \mu)$ for all $\wp, \tau, \mu \in [0,1]$;
5. If $\wp \leq \mu$ and $\tau \leq \mathfrak{d}$ with $\wp, \tau, \mu, \mathfrak{d} \in [0,1]$, then $\wp * \tau \leq \mu * \mathfrak{d}$.

Definition 2 [16] A binary operation $\circ: [0,1] \times [0,1] \to [0,1]$ is called a continuous t-conorm if:

1. $\wp \circ \tau = \tau \circ \wp$ for all $\wp, \tau \in [0,1]$;
2. $\circ$ is continuous;
3. $\wp \circ 0 = 0$ for all $\wp \in [0,1]$;
4. $(\wp \circ \tau) \circ \mu = \wp \circ (\tau \circ \mu)$ for all $\wp, \tau, \mu \in [0,1]$;
5. If $\wp \leq \mu$ and $\mu \leq \mathfrak{d}$ with $\wp, \tau, \mu, \mathfrak{d} \in [0,1]$, then $\wp \circ \tau \leq \mu \circ \mathfrak{d}$.

Definition 3 [28] Let $\Delta \neq \emptyset$ and $\wp, \Gamma, \eta\colon \Delta \times \Delta \to [1,+\infty)$ be given non-comparable functions, $*$ be a continuous t-norm, $\circ$ be a continuous t-co-norm and Ω, Φ, Λ be neutrosophic sets. $\Delta \times \Delta \times (0,+\infty)$ is said to be a neutrosophic rectangular triple controlled metric on Δ if for any $\psi, \lambda \in \Delta$ and all distinct $\upsilon, \Gamma \in \Delta \backslash \{\psi, \lambda\}$, the following conditions are satisfied:

1. $\Omega(\psi,\Gamma,\vartheta) + \Phi(\psi,\Gamma,\vartheta) + \Lambda(\psi,\Gamma,\vartheta) \leq 3$;
2. $\Omega(\psi,\Gamma,\vartheta) > 0$;
3. $\Omega(\psi,\Gamma,\vartheta) = 1$ for all $\vartheta > 0$ if and only if $\psi = \Gamma$;
4. $\Omega(\psi,\Gamma,\vartheta) = \Omega(\Gamma,\psi,\vartheta)$;
5. $\Omega(\psi,\lambda,\vartheta + \varpi + \varsigma) \geq \Omega(\psi,\Gamma,\frac{\vartheta}{\wp(\psi,\Gamma)}) * \Omega(\Gamma,\upsilon,\frac{\varpi}{\Gamma(\Gamma,\upsilon)}) * \Omega(\upsilon,\lambda,\frac{\varsigma}{\eta(\upsilon,\lambda)})$;
6. $\Omega(\psi,\Gamma,\cdot)\colon (0,+\infty) \to [0,1]$ is continuous and $\lim_{\vartheta\to+\infty} \Omega(\psi,\Gamma,\vartheta) = 1$;
7. $\Phi(\psi,\Gamma,\vartheta) < 1$;
8. $\Phi(\psi,\Gamma,\vartheta) = 0$ for all $\vartheta > 0$ if and only if $\psi = \Gamma$;
9. $\Phi(\psi,\Gamma,\vartheta) = \Phi(\Gamma,\psi,\vartheta)$;
10. $\Phi(\psi,\lambda,\vartheta + \varpi + \varsigma) \leq \Phi(\psi,\Gamma,\frac{\vartheta}{\wp(\psi,\Gamma)}) \circ \Phi(\Gamma,\upsilon,\frac{\varpi}{\Gamma(\Gamma,\upsilon)}) \circ \Phi(\upsilon,\lambda,\frac{\varsigma}{\eta(\upsilon,\lambda)})$;
11. $\Phi(\psi,\Gamma,\cdot)\colon (0,+\infty) \to [0,1]$ is continuous and $\lim_{\vartheta\to+\infty} \Phi(\psi,\Gamma,\vartheta) = 0$;
12. $\Lambda(\psi,\Gamma,\vartheta) < 1$;
13. $\Lambda(\psi,\Gamma,\vartheta) = 0$ for all $\vartheta > 0$ if and only if $\psi = \Gamma$;
14. $\Lambda(\psi,\Gamma,\vartheta) = \Lambda(\Gamma,\psi,\vartheta)$;
15. $\Lambda(\psi,\lambda,\vartheta + \varpi + \varsigma) \leq \Lambda(\psi,\Gamma,\frac{\vartheta}{\wp(\psi,\Gamma)}) \circ \Lambda(\Gamma,\upsilon,\frac{\varpi}{\Gamma(\Gamma,\upsilon)}) \circ \Lambda(\upsilon,\lambda,\frac{\varpi}{\eta(\upsilon,\lambda)})$;
16. $\Lambda(\psi,\Gamma,\cdot)\colon (0,+\infty) \to [0,1]$ is continuous and $\lim_{\vartheta\to+\infty} \Lambda(\psi,\Gamma,\vartheta) = 0$;
17. If $\vartheta \leq 0$, then $\Omega(\psi,\Gamma,\vartheta) = 0$, $\Phi(\psi,\Gamma,\vartheta) = 1$ and $\mathcal{S}(\psi,\Gamma,\vartheta) = 1$.

Then $(\Delta, \Omega, \Phi, \Lambda, *, \circ)$ is called a neutrosophic rectangular triple controlled metric space.

3 Main results

In this part, we present extended neutrosophic rectangular metric space and demonstrate some fixed point results.

Definition 4 Let $\Delta \neq \emptyset$ and $\wp\colon \Delta \times \Delta \to [1,+\infty)$ be given non-comparable functions, $*$ be a continuous t-norm, $\circ$ be a continuous t-co-norm and Ω, Φ, Λ be neutrosophic sets. $\Delta \times \Delta \times (0,+\infty)$ is said to be an extended neutrosophic rectangular metric on Δ if for any $\psi, \lambda \in \Delta$ and all distinct $\upsilon, \Gamma, \lambda \in \Delta$, the following conditions are satisfied:

(A1) $\Omega(\psi,\Gamma,\vartheta) + \Phi(\psi,\Gamma,\vartheta) + \Lambda(\psi,\Gamma,\vartheta) \leq 3$;
(A2) $\Omega(\psi,\Gamma,\vartheta) > 0$;
(A3) $\Omega(\psi,\Gamma,\vartheta) = 1$ for all $\vartheta > 0$ if and only if $\psi = \Gamma$;
(A4) $\Omega(\psi,\Gamma,\vartheta) = \Omega(\Gamma,\psi,\vartheta)$;
(A5) $\Omega(\psi,\lambda,\wp(\psi,\lambda)(\vartheta + \varpi + \varsigma)) \geq \Omega(\psi,\Gamma,\vartheta) * \Omega(\Gamma,\upsilon,\varpi) * \Omega(\upsilon,\lambda,\varsigma)$;
(A6) $\Omega(\psi,\Gamma,\cdot)\colon (0,+\infty) \to [0,1]$ is continuous and $\lim_{\vartheta\to+\infty} \Omega(\psi,\Gamma,\vartheta) = 1$;
(A7) $\Phi(\psi,\Gamma,\vartheta) < 1$;
(A8) $\Phi(\psi,\Gamma,\vartheta) = 0$ for all $\vartheta > 0$ if and only if $\psi = \Gamma$;
(A9) $\Phi(\psi,\Gamma,\vartheta) = \Phi(\Gamma,\psi,\vartheta)$;
(A10) $\Phi(\psi,\lambda,\wp(\psi,\lambda)(\vartheta + \varpi + \varsigma) \leq \Phi(\psi,\Gamma,\vartheta) \circ \Phi(\Gamma,\upsilon,\varpi) \circ \Phi(\upsilon,\lambda,\varsigma)$;
(A11) $\Phi(\psi,\Gamma,\cdot)\colon (0,+\infty) \to [0,1]$ is continuous and $\lim_{\vartheta\to+\infty} \Phi(\psi,\Gamma,\vartheta) = 0$;
(A12) $\Lambda(\psi,\Gamma,\vartheta) < 1$;
(A13) $\Lambda(\psi,\Gamma,\vartheta) = 0$ for all $\vartheta > 0$ if and only if $\psi = \Gamma$;
(A14) $\Lambda(\psi,\Gamma,\vartheta) = \Lambda(\Gamma,\psi,\vartheta)$;

(A15) $\Lambda(\psi,\lambda,\wp(\psi,\lambda)(\vartheta+\varpi+\varsigma)) \leq \Lambda(\psi,\Gamma,\vartheta) \circ \Lambda(\Gamma,\upsilon,\varpi) \circ \Lambda(\upsilon,\lambda,\varpi)$;

(A16) $\Lambda(\psi,\Gamma,\cdot)\colon (0,+\infty) \to [0,1]$ is continuous and $\lim_{\vartheta\to+\infty} \Lambda(\psi,\Gamma,\vartheta) = 0$;

(A17) If $\vartheta \leq 0$, then $\Omega(\psi,\Gamma,\vartheta) = 0$, $\Phi(\psi,\Gamma,\vartheta) = 1$ and $\mathcal{S}(\psi,\Gamma,\vartheta) = 1$.

Then $(\Delta,\Omega,\Phi,\Lambda,*,\circ)$ is called an extended neutrosophic rectangular metric space (ENRMS).

Example 1 Let $\Delta = \{1,2,3,4\}$ and $\wp\colon \Delta\times\Delta \to [1,+\infty)$ be a function given by $\wp(\psi,\Gamma) = \psi+\Gamma+1$. Define $\Omega,\Phi,\Lambda\colon \Delta\times\Delta\times(0,+\infty) \to [0,1]$ as

$$\Omega(\psi,\Gamma,\vartheta) = \begin{cases} 1, & \text{if } \psi=\Gamma \\ \frac{\vartheta}{\vartheta+\max\{\psi,\Gamma\}^2}, & \text{if otherwise,} \end{cases}$$

$$\Phi(\psi,\Gamma,\vartheta) = \begin{cases} 0, & \text{if } \psi=\Gamma \\ \frac{\max\{\psi,\Gamma\}^2}{\vartheta+\max\{\psi,\Gamma\}^2}, & \text{if otherwise,} \end{cases}$$

and

$$\Lambda(\psi,\Gamma,\vartheta) = \begin{cases} 0, & \text{if } \psi=\Gamma \\ \frac{\max\{\psi,\Gamma\}^2}{\vartheta}, & \text{if otherwise.} \end{cases}$$

Then $(\Delta,\Omega,\Phi,\Lambda,*,\circ)$ is an ENRMS with continuous t-norm $\wp * \tau = \wp\tau$ and continuous t-co-norm, $\wp \circ \bar{a} = \max\{\wp,\bar{a}\}$.

Here we prove (A5), (A10) and (A15), others are obvious.

Let $\psi=1$, $\Gamma=2$, $\upsilon=3$ and $\lambda=4$. Then

$$\Omega(1,4,\vartheta+\varpi+\varsigma) = \frac{\vartheta+\varpi+\varsigma}{\vartheta+\varpi+\varsigma+\max\{1,4\}^2} = \frac{\vartheta+\varpi+\varsigma}{\vartheta+\varpi+\varsigma+16}.$$

On the other hand,

$$\Omega\left(1,2,\frac{\vartheta}{\wp(1,4)}\right) = \frac{\frac{\vartheta}{\wp(1,4)}}{\frac{\vartheta}{\wp(1,4)}+\max\{1,2\}^2} = \frac{\frac{\vartheta}{6}}{\frac{\vartheta}{6}+4} = \frac{\vartheta}{\vartheta+24},$$

$$\Omega\left(2,3,\frac{\varpi}{\wp(1,4)}\right) = \frac{\frac{\varpi}{\wp(1,4)}}{\frac{\varpi}{\wp(1,4)}+\max\{2,3\}^2} = \frac{\frac{\varpi}{6}}{\frac{\varpi}{6}+9} = \frac{\varpi}{\varpi+54}$$

and

$$\Omega\left(3,4,\frac{\varsigma}{\wp(1,4)}\right) = \frac{\frac{\varsigma}{\wp(1,4)}}{\frac{\varsigma}{\wp(1,4)}+\max\{3,4\}^2} = \frac{\frac{\varsigma}{6}}{\frac{\varsigma}{6}+16} = \frac{\varsigma}{\varsigma+96}.$$

That is,

$$\frac{\vartheta+\varpi+\varsigma}{\vartheta+\varpi+\varsigma+16} \geq \frac{\vartheta}{\vartheta+24}\cdot\frac{\varpi}{\varpi+54}.\frac{\varsigma}{\varsigma+96}.$$

Then it satisfies all $\vartheta,\varpi,\varsigma > 0$. Hence,

$$\Omega(\psi,\lambda,\vartheta+\varpi+\varsigma) \geq \Omega\left(\psi,\Gamma,\frac{\vartheta}{\wp(\psi,\lambda)}\right) * \Omega\left(\Gamma,\upsilon,\frac{\varpi}{\wp(\psi,\lambda)}\right) * \Omega\left(\upsilon,\lambda,\frac{\varsigma}{\wp(\psi,\lambda)}\right).$$

Now,

$$\Phi(1,4,\vartheta+\varpi+\varsigma)=\frac{\max\{1,4\}^2}{\vartheta+\varpi+\varsigma+\max\{1,4\}^2}=\frac{16}{\vartheta+\varpi+\varsigma+16}.$$

On the other hand,

$$\Phi\left(1,2,\frac{\vartheta}{\wp(1,4)}\right)=\frac{\max\{1,2\}^2}{\frac{\vartheta}{\wp(1,4)}+\max\{1,2\}^2}=\frac{4}{\frac{\vartheta}{6}+4}=\frac{24}{\vartheta+24},$$
$$\Phi\left(2,3,\frac{\varpi}{\wp(1,4)}\right)=\frac{\max\{2,3\}^2}{\frac{\varpi}{\wp(1,4)}+\max\{2,3\}^2}=\frac{9}{\frac{\varpi}{6}+9}=\frac{54}{\varpi+54}$$

and

$$\Phi\left(3,4,\frac{\varsigma}{\wp(1,4)}\right)=\frac{\max\{3,4\}^2}{\frac{\varsigma}{\wp(1,4)}+\max\{3,4\}^2}=\frac{16}{\frac{\varsigma}{6}+16}=\frac{96}{\varsigma+96}.$$

That is,

$$\frac{16}{\vartheta+\varpi+\varsigma+16}\leq\max\left\{\frac{24}{\vartheta+24},\frac{54}{\varpi+54},\frac{96}{\varsigma+96}\right\}.$$

Then it satisfies all $\vartheta,\varpi,\varsigma>0$. Hence,

$$\Phi(\psi,\lambda,\vartheta+\varpi+\varsigma)\leq\Phi\left(\psi,\Gamma,\frac{\vartheta}{\wp(\psi,\lambda)}\right)\circ\Phi\left(\upsilon,\lambda,\frac{\varpi}{\wp(\psi,\lambda)}\right)\circ\Phi\left(\upsilon,\lambda,\frac{\varsigma}{\wp(\psi,\lambda)}\right).$$

Now,

$$\Lambda(1,3,\vartheta+\varpi+\varsigma)=\frac{\max\{1,3\}^2}{\vartheta+\varpi+\varsigma}=\frac{9}{\vartheta+\varpi+\varsigma}.$$

On the other hand,

$$\Lambda\left(1,2,\frac{\vartheta}{\wp(1,4)}\right)=\frac{\max\{1,2\}^2}{\frac{\vartheta}{\wp(1,4)}}=\frac{4}{\frac{\vartheta}{6}}=\frac{24}{\vartheta},$$
$$\Lambda\left(2,3,\frac{\varpi}{\wp(1,4)}\right)=\frac{\max\{2,3\}^2}{\frac{\varpi}{\wp(1,4)}}=\frac{9}{\frac{\varpi}{6}}=\frac{54}{\varpi}$$

and

$$\Lambda\left(3,4,\frac{\varsigma}{\wp(1,4)}\right)=\frac{\max\{3,4\}^2}{\frac{\varsigma}{\wp(1,4)}}=\frac{16}{\frac{\varsigma}{6}}=\frac{96}{\varsigma}.$$

That is,

$$\frac{9}{\vartheta+\varpi+\varsigma}\leq\max\left\{\frac{24}{\vartheta},\frac{54}{\varpi},\frac{96}{\varsigma}\right\}.$$

Then it satisfies all $\vartheta, \varpi > 0$. Hence,

$$\Lambda(\psi,\lambda,\vartheta+\varpi+\varsigma) \leq \Lambda\left(\psi,\Gamma,\frac{\vartheta}{\wp(\psi,\lambda)}\right) \circ \Lambda\left(\Gamma,\upsilon,\frac{\varpi}{\wp(\psi,\lambda)}\right) \circ \Lambda\left(\upsilon,\lambda,\frac{\varpi}{\wp(\psi,\lambda)}\right).$$

Hence $(\Delta,\Omega,\Phi,\Lambda,*,\circ)$ is an ENRMS.

Remark 1 The preceding example also satisfies for continuous t-norm $\wp * \bar{a} = \min\{\wp,\bar{a}\}$ and continuous t-co-norm $\wp \circ \bar{a} = \max\{\wp,\bar{a}\}$.

Definition 5 Let $(\Delta,\Omega,\Phi,\Lambda,*,\circ)$ be an ENRMS, an open ball is then defined $\Lambda(\psi,\varrho,\vartheta)$ with centre ψ, radius $\varrho, 0<\varrho<1$ and $\vartheta>0$ as follows:

$$\Lambda(\psi,\varrho,\vartheta) = \left\{\Gamma \in \Delta \colon \Omega(\psi,\Gamma,\vartheta) > 1-\varrho, \Phi(\psi,\Gamma,\vartheta) < \varrho, \Lambda(\psi,\Gamma,\vartheta) < \varrho\right\}.$$

Definition 6 Let $(\Delta,\Omega,\Phi,\Lambda,*,\circ)$ be an ENRMS and $\{\psi_\kappa\}$ be a sequence in Δ. Then $\{\psi_\kappa\}$ is said to be:

1. Convergent if there exists $\psi \in \Delta$ such that

$$\lim_{\kappa\to+\infty} \Omega(\psi_\kappa,\psi,\vartheta) = 1, \qquad \lim_{\kappa\to+\infty} \Phi(\psi_\kappa,\psi,\vartheta) = 0,$$
$$\lim_{\kappa\to+\infty} \Lambda(\psi_\kappa,\psi,\vartheta) = 0 \quad \text{for all } \vartheta > 0;$$

2. Cauchy sequence if and only if for each $\bar{a}>0$, $\vartheta>0$, there exists $\kappa_0 \in \mathbb{N}$ such that

$$\Omega(\psi_\kappa,\psi_{\kappa+\omega},\vartheta) \geq 1-\bar{a}, \qquad \Phi(\psi_\kappa,\psi_{\kappa+\omega},\vartheta) \leq \bar{a}, \qquad \Phi(\psi_\kappa,\psi_{\kappa+\omega},\vartheta) \leq \bar{a}$$

for all $\kappa,\pi \geq \kappa_0$.

If every Cauchy sequence is convergent in Δ, then $(\Delta,\Omega,\Phi,\Lambda,*,\circ)$ is called a complete ENRMS.

Lemma 1 *Let $\{\psi_\kappa\}$ be a Cauchy sequence in ENRMS $(\Delta,\Omega,\Phi,\Lambda,*,\circ)$ such that $\psi_\kappa \neq \psi_\pi$ whenever $\pi,\kappa \in \mathbb{N}$ with $\kappa \neq \pi$. Then the sequence $\{\psi_\kappa\}$ can converge to, at most, one limit point.*

Proof Contrarily, assume that $\psi_\kappa \to \psi$ and $\psi_\kappa \to \Gamma$ for $\psi \neq \Gamma$. Then

$$\lim_{\kappa\to+\infty} \Omega(\psi_\kappa,\psi,\vartheta) = 1, \qquad \lim_{\kappa\to+\infty} \Phi(\psi_\kappa,\psi,\vartheta) = 0, \qquad \lim_{\kappa\to+\infty} \Lambda(\psi_\kappa,\psi,\vartheta) = 0$$

and

$$\lim_{\kappa\to+\infty} \Omega(\psi_\kappa,\Gamma,\vartheta) = 1, \qquad \lim_{\kappa\to+\infty} \Phi(\psi_\kappa,\Gamma,\vartheta) = 0, \qquad \lim_{\kappa\to+\infty} \Lambda(\psi_\kappa,\Gamma,\vartheta) = 0$$

for all $\vartheta > 0$. Suppose

$$\begin{aligned}\Omega(\psi,\Gamma,\vartheta) \geq{}& \Omega\left(\psi,\psi_\kappa,\frac{\vartheta}{3\wp(\psi,\Gamma)}\right) * \Omega\left(\psi_\kappa,\psi_{\kappa+1},\frac{\vartheta}{3\wp(\psi,\Gamma)}\right)\\ &* \Omega\left(\psi_{\kappa+1},\Gamma,\frac{\vartheta}{3\wp(\psi,\Gamma)}\right)\end{aligned}$$

$$
\begin{aligned}
&\to 1 * 1 * 1 \quad \text{as } \kappa \to +\infty, \\
\Phi(\psi, \Gamma, \vartheta) \le{}& \Phi\left(\psi, \psi_\kappa, \frac{\vartheta}{3\wp(\psi, \Gamma)}\right) \circ \Phi\left(\psi_\kappa, \psi_{\kappa+1}, \frac{\vartheta}{3\wp(\psi, \Gamma)}\right) \\
&\circ \Phi\left(\psi_{\kappa+1}, \Gamma, \frac{\vartheta}{3\wp(\psi, \Gamma)}\right) \\
&\to 0 \circ 0 \circ 0 \quad \text{as } \kappa \to +\infty, \\
\Lambda(\psi, \Gamma, \vartheta) \le{}& \Lambda\left(\psi, \psi_\kappa, \frac{\vartheta}{3\wp(\psi, \Gamma)}\right) \circ \Lambda\left(\psi_\kappa, \psi_{\kappa+1}, \frac{\vartheta}{3\wp(\psi, \Gamma)}\right) \\
&\circ \Lambda\left(\psi_{\kappa+1}, \Gamma, \frac{\vartheta}{3\wp(\psi, \Gamma)}\right) \\
&\to 0 \circ 0 \circ 0 \quad \text{as } \kappa \to +\infty.
\end{aligned}
$$

That is, $\Omega(\psi, \Gamma, \vartheta) \ge 1 * 1 * 1 = 1$, $\Phi(\psi, \Gamma, \vartheta) \le 0 \circ 0 \circ 0 = 0$ and $\Lambda(\psi, \Gamma, \vartheta) \le 0 \circ 0 \circ 0 = 0$. Hence $\psi = \Gamma$, that is, the sequence converges to at most one limit point. □

Lemma 2 *Let* $(\Delta, \Omega, \Phi, \Lambda, *, \circ)$ *be an ENRMS. If for some* $0 < \theta < 1$ *and for any* $\psi, \Gamma \in \Delta$, $\vartheta > 0$,

$$
\begin{aligned}
&\Omega(\psi, \Gamma, \vartheta) \ge \Omega\left(\psi, \Gamma, \frac{\vartheta}{\theta}\right), \qquad \Phi(\psi, \Gamma, \vartheta) \le \Phi\left(\psi, \Gamma, \frac{\vartheta}{\theta}\right), \\
&\Lambda(\psi, \Gamma, \vartheta) \le \Lambda\left(\psi, \Gamma, \frac{\vartheta}{\theta}\right),
\end{aligned}
\tag{1}
$$

then $\psi = \Gamma$.

Proof Condition (1) implies that

$$
\begin{aligned}
&\Omega(\psi, \Gamma, \vartheta) \ge \Omega\left(\psi, \Gamma, \frac{\vartheta}{\theta^\kappa}\right), \qquad \Phi(\psi, \Gamma, \vartheta) \le \Phi\left(\psi, \Gamma, \frac{\vartheta}{\theta^\kappa}\right), \\
&\Lambda(\psi, \Gamma, \vartheta) \le \Lambda\left(\psi, \Gamma, \frac{\vartheta}{\theta^\kappa}\right),
\end{aligned}
$$

$\kappa \in \mathbb{N}$, $\vartheta > 0$.

Now, we have

$$
\begin{aligned}
&\Omega(\psi, \Gamma, \vartheta) \ge \lim_{\kappa \to +\infty} \Omega\left(\psi, \Gamma, \frac{\vartheta}{\theta^\kappa}\right) = 1, \\
&\Phi(\psi, \Gamma, \vartheta) \le \lim_{\kappa \to +\infty} \Phi\left(\psi, \Gamma, \frac{\vartheta}{\theta^\kappa}\right) = 0, \\
&\Lambda(\psi, \Gamma, \vartheta) \le \lim_{\kappa \to +\infty} \Lambda\left(\psi, \Gamma, \frac{\vartheta}{\theta^\kappa}\right) = 0, \quad \vartheta > 0.
\end{aligned}
$$

Also, by Definition 4 of (A3), (A8), (A13), we obtain $\psi = \Gamma$. □

Theorem 1 *Suppose that* $(\Delta, \Omega, \Phi, \Lambda, *, \circ)$ *is a complete ENRMS in the company of* $\wp\colon \Delta \times \Delta \to [1, +\infty)$ *with* $0 < \theta < 1$ *and suppose that*

$$\begin{aligned} &\lim_{\vartheta \to +\infty} \Omega(\psi, \Gamma, \vartheta) = 1, \quad \lim_{\vartheta \to +\infty} \Phi(\psi, \Gamma, \vartheta) = 0 \quad \textit{and} \\ &\lim_{\vartheta \to +\infty} \Lambda(\psi, \Gamma, \vartheta) = 0 \end{aligned} \tag{2}$$

for all $\psi, \Gamma \in \Delta$ *and* $\vartheta > 0$. *Let* $\nabla\colon \Delta \to \Delta$ *be a mapping satisfying*

$$\begin{aligned} &\Omega(\nabla\psi, \nabla\Gamma, \theta\vartheta) \ge \Omega(\psi, \Gamma, \vartheta), \\ &\Phi(\nabla\psi, \nabla\Gamma, \theta\vartheta) \le \Phi(\psi, \Gamma, \vartheta) \quad \textit{and} \quad \Lambda(\nabla\psi, \nabla\Gamma, \theta\vartheta) \le \Lambda(\psi, \Gamma, \vartheta) \end{aligned} \tag{3}$$

for all $\psi, \Gamma \in \Delta$ *and* $\vartheta > 0$. *Further, suppose that for arbitrary* $\psi_0 \in \Delta$ *and* $\kappa, \omega \in \mathbb{N}$, *we have*

$$\wp(\psi_\kappa, \psi_{\kappa+\omega}) < \frac{1}{\theta}.$$

Then ∇ *has a unique fixed point.*

Proof Let $\psi_0 \in \Delta$ and define a sequence ψ_κ by $\psi_\kappa = \nabla^\kappa \psi_0 = \nabla\psi_{\kappa-1}$, $\kappa \in \mathbb{N}$.

By utilising (2) for all $\vartheta > 0$, we obtain

$$\begin{aligned} \Omega(\psi_\kappa, \psi_{\kappa+1}, \theta\vartheta) &= \Omega(\nabla\psi_{\kappa-1}, \nabla\psi_\kappa, \theta\vartheta) \ge \Omega(\psi_{\kappa-1}, \psi_\kappa, \vartheta) \ge \Omega\left(\psi_{\kappa-2}, \psi_{\kappa-1}, \frac{\vartheta}{\theta}\right) \\ &\ge \Omega\left(\psi_{\kappa-3}, \psi_{\kappa-2}, \frac{\vartheta}{\theta^2}\right) \ge \cdots \ge \Omega\left(\psi_0, \psi_1, \frac{\vartheta}{\theta^{\kappa-1}}\right), \\ \Phi(\psi_\kappa, \psi_{\kappa+1}, \theta\vartheta) &= \Phi(\nabla\psi_{\kappa-1}, \nabla\psi_\kappa, \theta\vartheta) \le \Phi(\psi_{\kappa-1}, \psi_\kappa, \vartheta) \le \Phi\left(\psi_{\kappa-2}, \psi_{\kappa-1}, \frac{\vartheta}{\theta}\right) \\ &\le \Phi\left(\psi_{\kappa-3}, \psi_{\kappa-2}, \frac{\vartheta}{\theta^2}\right) \le \cdots \le \Phi\left(\psi_0, \psi_1, \frac{\vartheta}{\theta^{\kappa-1}}\right) \end{aligned}$$

and

$$\begin{aligned} \Lambda(\psi_\kappa, \psi_{\kappa+1}, \theta\vartheta) &= \Lambda(\nabla\psi_{\kappa-1}, \nabla\psi_\kappa, \vartheta) \le \Lambda(\psi_{\kappa-1}, \psi_\kappa, \vartheta) \le \Lambda\left(\psi_{\kappa-2}, \psi_{\kappa-1}, \frac{\vartheta}{\theta}\right) \\ &\le \Lambda\left(\psi_{\kappa-3}, \psi_{\kappa-2}, \frac{\vartheta}{\theta^2}\right) \le \cdots \le \Lambda\left(\psi_0, \psi_1, \frac{\vartheta}{\theta^{\kappa-1}}\right). \end{aligned}$$

We obtain

$$\begin{aligned} &\Omega(\psi_\kappa, \psi_{\kappa+1}, \theta\vartheta) \ge \Omega\left(\psi_0, \psi_1, \frac{\vartheta}{\theta^{\kappa-1}}\right), \\ &\Phi(\psi_\kappa, \psi_{\kappa+1}, \theta\vartheta) \le \Phi\left(\psi_0, \psi_1, \frac{\vartheta}{\theta^{\kappa-1}}\right) \quad \text{and} \\ &\Lambda(\psi_\kappa, \psi_{\kappa+1}, \theta\vartheta) \le \Lambda\left(\psi_0, \psi_1, \frac{\vartheta}{\theta^{\kappa-1}}\right). \end{aligned} \tag{4}$$

Using (A5), (A10) and (A15), we have the following cases:

Case 1. When $\mathrm{i} = 2\pi + 1$, i.e. i is odd, then

$$
\begin{aligned}
&\Omega(\psi_\kappa, \psi_{\kappa+2\pi+1}, \vartheta) \\
&\quad \geq \Omega\left(\psi_\kappa, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\
&\qquad * \Omega\left(\psi_{\kappa+1}, \psi_{\kappa+2}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\
&\qquad * \cdots \\
&\qquad * \Omega\left(\psi_{\kappa+2\pi-1}, \psi_{\kappa+2\pi}, \frac{\vartheta}{3^\pi(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\
&\qquad * \Omega\left(\psi_{\kappa+2\pi}, \psi_{\kappa+2\pi+1}, \frac{\vartheta}{3^\pi(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right), \\
&\Phi(\psi_\kappa, \psi_{\kappa+2\pi+1}, \vartheta) \\
&\quad \leq \Phi\left(\psi_\kappa, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\
&\qquad \circ \Phi\left(\psi_{\kappa+1}, \psi_{\kappa+2}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\
&\qquad \circ \cdots \\
&\qquad \circ \Phi\left(\psi_{\kappa+2\pi-1}, \psi_{\kappa+2\pi}, \frac{\vartheta}{3^\pi(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\
&\qquad \circ \Phi\left(\psi_{\kappa+2\pi}, \psi_{\kappa+2\pi+1}, \frac{\vartheta}{3^\pi(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Lambda(\psi_\kappa, \psi_{\kappa+2\pi+1}, \vartheta) \\
&\quad \leq \Lambda\left(\psi_\kappa, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\
&\qquad \circ \Lambda\left(\psi_{\kappa+1}, \psi_{\kappa+2}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\
&\qquad \circ \cdots \\
&\qquad \circ \Lambda\left(\psi_{\kappa+2\pi-1}, \psi_{\kappa+2\pi}, \frac{\vartheta}{3^\pi(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\
&\qquad \circ \Lambda\left(\psi_{\kappa+2\pi}, \psi_{\kappa+2\pi+1}, \frac{\vartheta}{3^\pi(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right).
\end{aligned}
$$

Using (4) in the above inequalities, we deduce

$$
\begin{aligned}
&\Omega(\psi_\kappa, \psi_{\kappa+2\pi+1}, \vartheta) \\
&\quad \geq \Omega\left(\psi_0, \psi_1, \frac{\vartheta}{3\theta^{\kappa-1}(\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\
&\qquad * \Omega\left(\psi_0, \psi_1, \frac{\vartheta}{3\theta^{\kappa}(\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad * \cdots \\
&\quad * \Omega\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi}\theta^{\kappa+2\pi-2}(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right) \\
&\quad * \Omega\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi}\theta^{\kappa+2\pi-1}(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right), \\
&\Phi(\psi_{\kappa}, \psi_{\kappa+2\pi+1}, \vartheta) \\
&\quad \leq \Phi\left(\psi_0, \psi_1, \frac{\vartheta}{3\theta^{\kappa-1}(\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right) \\
&\quad \circ \Phi\left(\psi_0, \psi_1, \frac{\vartheta}{3\theta^{\kappa}(\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right) \\
&\quad \circ \cdots \\
&\quad \circ \Phi\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi}\theta^{\kappa+2\pi-2}(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right) \\
&\quad \circ \Phi\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi}\theta^{\kappa+2\pi-1}(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right), \\
&\Lambda(\psi_{\kappa}, \psi_{\kappa+2\pi+1}, \vartheta) \\
&\quad \leq \Lambda\left(\psi_0, \psi_1, \frac{\vartheta}{3\theta^{\kappa-1}(\wp(\psi_{\kappa}, \psi_{\kappa+1}))}\right) \circ \Lambda\left(\psi_0, \psi_1, \frac{\vartheta}{3\theta^{\kappa}(\wp(\psi_{\kappa+1}, \psi_{\kappa+2}))}\right) \\
&\quad \circ \cdots \\
&\quad \circ \Lambda\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi}\theta^{\kappa+2\pi-2}(\wp(\psi_{\kappa+2\pi-1}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa+2}, \psi_{\kappa+2\pi+1}))}\right) \\
&\quad \circ \Lambda\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi}\theta^{\kappa+2\pi-1}(\wp(\psi_{\kappa+2\pi}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa+2}, \psi_{\kappa+2\pi+1}))}\right).
\end{aligned}
$$

Case 2. When $\mathfrak{i} = 2\pi$, i.e. $\mathfrak{i}$ is even, then

$$
\begin{aligned}
&\Omega(\psi_{\kappa}, \psi_{\kappa+2\pi}, \vartheta) \\
&\quad \geq \Omega\left(\psi_{\kappa}, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi_{\kappa}, \psi_{\kappa+2\pi}))}\right) * \Omega\left(\psi_{\kappa+1}, \psi_{\kappa+2}, \frac{\vartheta}{3(\wp(\psi_{\kappa}, \psi_{\kappa+2\pi}))}\right) \\
&\quad * \cdots \\
&\quad * \Omega\left(\psi_{\kappa+2\pi-3}, \psi_{\kappa+2\pi-2}, \frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi})\cdots\wp(\psi_{\kappa}, \psi_{\kappa+2\pi}))}\right) \\
&\quad * \Omega\left(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi}, \frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi})\cdots\wp(\psi_{\kappa}, \psi_{\kappa+2\pi}))}\right), \\
&\Phi(\psi_{\kappa}, \psi_{\kappa+2\pi}, \vartheta) \\
&\quad \leq \Phi\left(\psi_{\kappa}, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi_{\kappa}, \psi_{\kappa+2\pi}))}\right) \circ \Phi\left(\psi_{\kappa+1}, \psi_{\kappa+2}, \frac{\vartheta}{3(\wp(\psi_{\kappa}, \psi_{\kappa+2\pi}))}\right) \\
&\quad \circ \cdots \\
&\quad \circ \Phi\left(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi}, \frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi})\cdots\wp(\psi_{\kappa}, \psi_{\kappa+2\pi}))}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Lambda(\psi_\kappa, \psi_{\kappa+2\pi}, \vartheta) \\
&\quad \le \Lambda\left(\psi_\kappa, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \circ \Lambda\left(\psi_{\kappa+1}, \psi_{\kappa+2}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \\
&\qquad \circ \cdots \\
&\qquad \circ \Lambda\left(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi-3}, \frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \\
&\qquad \circ \Lambda\left(\psi_{\kappa+2\pi-3}, \psi_{\kappa+2\pi-2}, \frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \\
&\qquad \circ \Lambda\left(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi}, \frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right).
\end{aligned}
$$

Using (4) in the above inequalities, we deduce

$$
\begin{aligned}
&\Omega(\psi_\kappa, \psi_{\kappa+2\pi}, \vartheta) \\
&\quad \ge \Omega\left(\psi_0, \psi_1, \frac{\vartheta}{3\theta^{\kappa-1}(\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) * \Omega\left(\psi_0, \psi_1, \frac{\vartheta}{3\theta^{\kappa}(\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \\
&\qquad * \cdots \\
&\qquad * \Omega\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi-1}\theta^{\kappa+2\pi-5}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \\
&\qquad * \Omega\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi-1}\theta^{\kappa+2\pi-4}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \\
&\qquad * \Omega\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi-1}\theta^{\kappa+2\pi-3}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right), \\
&\Phi(\psi_\kappa, \psi_{\kappa+2\pi}, \vartheta) \\
&\quad \le \Phi\left(\psi_0, \psi_1, \frac{\vartheta}{3\theta^{\kappa-1}(\wp(\psi_\kappa, \psi_{\kappa+1}))}\right) \circ \Phi\left(\psi_{\kappa+1}, \psi_{\kappa+2}, \frac{\vartheta}{3\theta^{\kappa}(\wp(\psi_{\kappa+1}, \psi_{\kappa+2}))}\right) \\
&\qquad \circ \cdots \\
&\qquad \circ \Phi\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi-1}\theta^{\kappa+2\pi-5}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi-3})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\cdots\wp(\psi_{\kappa+2}, \psi_{\kappa+2\pi}))}\right) \\
&\qquad \circ \Phi\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi-1}\theta^{\kappa+2\pi-4}(\wp(\psi_{\kappa+2\pi-3}, \psi_{\kappa+2\pi-2})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\cdots\wp(\psi_{\kappa+2}, \psi_{\kappa+2\pi}))}\right) \\
&\qquad \circ \Phi\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi-1}\theta^{\kappa+2\pi-3}(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\cdots\wp(\psi_{\kappa+2}, \psi_{\kappa+2\pi}))}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Lambda(\psi_\kappa, \psi_{\kappa+2\pi}, \vartheta) \\
&\quad \le \Lambda\left(\psi_0, \psi_1, \frac{\vartheta}{3\theta^{\kappa-1}(\wp(\psi_\kappa, \psi_{\kappa+1}))}\right) \circ \Lambda\left(\psi_{\kappa+1}, \psi_{\kappa+2}, \frac{\vartheta}{3\theta^{\kappa}(\wp(\psi_{\kappa+1}, \psi_{\kappa+2}))}\right) \\
&\qquad \circ \cdots \\
&\qquad \circ \Lambda\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi-1}\theta^{\kappa+2\pi-5}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi-3})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\cdots\wp(\psi_{\kappa+2}, \psi_{\kappa+2\pi}))}\right)
\end{aligned}
$$

$$\circ \Lambda\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi-1}\theta^{\kappa+2\pi-4}(\wp(\psi_{\kappa+2\pi-3}, \psi_{\kappa+2\pi-2})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\cdots\wp(\psi_{\kappa+2}, \psi_{\kappa+2\pi}))}\right)$$
$$\circ \Lambda\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi-1}\theta^{\kappa+2\pi-3}(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\cdots\wp(\psi_{\kappa+2}, \psi_{\kappa+2\pi}))}\right).$$

Since $\kappa, \omega \in \mathbb{N}$, we have

$$\wp(\psi_\kappa, \psi_{\kappa+\omega}) < \frac{1}{\theta}.$$

Therefore, from (2), for each case $\kappa \to +\infty$, we deduce

$$\lim_{\kappa\to+\infty} \Omega(\psi_\kappa, \psi_{\kappa+\mathrm{i}}, \vartheta) = 1 * 1 * \cdots * 1 = 1,$$
$$\lim_{\kappa\to+\infty} \Phi(\psi_\kappa, \psi_{\kappa+\mathrm{i}}, \vartheta) = 0 \circ 0 \circ \cdots \circ 0 = 0$$

and

$$\lim_{\kappa\to+\infty} \Lambda(\psi_\kappa, \psi_{\kappa+\mathrm{i}}, \vartheta) = 0 \circ 0 \circ \cdots \circ 0 = 0.$$

Therefore, $\{\psi_\kappa\}$ is a Cauchy sequence. Since $(\Delta, \Omega, \Phi, \Lambda, *, \circ)$ is complete, there exists

$$\lim_{\kappa\to+\infty} \psi_\kappa = \psi.$$

Using (A5), (A10), (A15) and (2), we get

$$\begin{aligned}
&\Omega(\psi, \nabla\psi, \vartheta) \\
&\quad \geq \Omega\left(\psi, \psi_\kappa, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) * \Omega\left(\psi_\kappa, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\qquad * \Omega\left(\psi_{\kappa+1}, \nabla\psi, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\quad = \Omega\left(\psi, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) * \Omega\left(\nabla\psi_{\kappa-1}, \nabla\psi_\kappa, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\qquad * \Omega\left(\nabla\psi_\kappa, \nabla\psi, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\quad \geq \Omega\left(\psi, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) * \Omega\left(\psi_{\kappa-1}, \psi_\kappa, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\qquad * \Omega\left(\psi_\kappa, \psi, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\quad \to 1 * 1 * 1 = 1 \quad \text{as } \kappa \to +\infty, \\
&\Phi(\psi, \nabla\psi, \vartheta) \\
&\quad \leq \Phi\left(\psi, \psi_\kappa, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \circ \Phi\left(\psi_\kappa, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\qquad \circ \Phi\left(\psi_{\kappa+1}, \nabla\psi, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \Phi\left(\psi, \psi_\kappa, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \circ \Phi\left(\nabla\psi_{\kappa-1}, \nabla\psi_\kappa, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\quad \circ \Phi\left(\nabla\psi_\kappa, \nabla\psi, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\le \Phi\left(\psi, \psi_\kappa, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \circ \Phi\left(\psi_{\kappa-1}, \psi_\kappa, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\quad \circ \Phi\left(\psi_\kappa, \psi, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\to 0 \circ 0 \circ 0 = 0 \quad \text{as } \kappa \to +\infty
\end{aligned}
$$

and

$$
\begin{aligned}
&\Lambda(\psi, \nabla\psi, \vartheta) \\
&\quad \le \Lambda\left(\psi, \psi_\kappa, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \circ \Lambda\left(\psi_\kappa, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\qquad \circ \Lambda\left(\psi_{\kappa+1}, \nabla\psi, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\quad = \Lambda\left(\psi, \psi_\kappa, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \circ \Lambda\left(\nabla\psi_{\kappa-1}, \nabla\psi_\kappa, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\qquad \circ \Lambda\left(\nabla\psi_\kappa, \nabla\psi, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\quad \le \Lambda\left(\psi, \psi_\kappa, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \circ \Lambda\left(\psi_{\kappa-1}, \psi_\kappa, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\qquad \circ \Lambda\left(\psi_\kappa, \psi, \frac{\vartheta}{3(\wp(\psi, \nabla\psi))}\right) \\
&\quad \to 0 \circ 0 \circ 0 = 0 \quad \text{as } \kappa \to +\infty.
\end{aligned}
$$

Hence, $\nabla\psi = \psi$. Let $\nabla\mu = \mu$ for some $\mu \in \Delta$, then

$$
\begin{aligned}
&1 \ge \Omega(\mu, \psi, \vartheta) = \Omega(\nabla\mu, \nabla\psi, \vartheta) \ge \Omega\left(\mu, \psi, \frac{\vartheta}{\theta}\right) = \Omega\left(\nabla\mu, \nabla\psi, \frac{\vartheta}{\theta}\right) \\
&\quad \ge \Omega\left(\mu, \psi, \frac{\vartheta}{\theta^2}\right) \ge \cdots \ge \Omega\left(\mu, \psi, \frac{\vartheta}{\theta^\kappa}\right) \to 1 \quad \text{as } \kappa \to +\infty, \\
&0 \le \Phi(\mu, \psi, \vartheta) = \Phi(\nabla\mu, \nabla\psi, \vartheta) \le \Phi\left(\mu, \psi, \frac{\vartheta}{\theta}\right) = \Phi\left(\nabla\mu, \nabla\psi, \frac{\vartheta}{\theta}\right) \\
&\quad \le \Phi\left(\mu, \psi, \frac{\vartheta}{\theta^2}\right) \le \cdots \le \Phi\left(\mu, \psi, \frac{\vartheta}{\theta^\kappa}\right) \to 0 \quad \text{as } \kappa \to +\infty
\end{aligned}
$$

and

$$
\begin{aligned}
&0 \le \Lambda(\mu, \psi, \vartheta) = \Lambda(\nabla\mu, \nabla\psi, \vartheta) \le \Lambda\left(\mu, \psi, \frac{\vartheta}{\theta}\right) = \Lambda\left(\nabla\mu, \nabla\psi, \frac{\vartheta}{\theta}\right) \\
&\quad \le \Lambda\left(\mu, \psi, \frac{\vartheta}{\theta^2}\right) \le \cdots \le \Lambda\left(\mu, \psi, \frac{\vartheta}{\theta^\kappa}\right) \to 0 \quad \text{as } \kappa \to +\infty
\end{aligned}
$$

by using (A3), (A8) and (A13), $\psi = \mu$. Therefore, ∇ has a unique fixed point. □

Definition 7 Let $(\Delta, \Omega, \Phi, \Lambda, *, \circ)$ be an ENRMS. A map $\nabla: \Delta \to \Delta$ is an ENRC(extended neutrosophic rectangular contraction) if there exists $0 < \theta < 1$ such that

$$\frac{1}{\Omega(\mathcal{P}\psi, \mathcal{P}\Gamma, \vartheta)} - 1 \le \theta\left[\frac{1}{\Omega(\psi, \Gamma, \vartheta)} - 1\right] \tag{5}$$

$$\Phi(\mathcal{P}\psi, \mathcal{P}\Gamma, \vartheta) \le \theta\Phi(\psi, \Gamma, \vartheta) \tag{6}$$

and

$$\Lambda(\mathcal{P}\psi, \mathcal{P}\Gamma, \vartheta) \le \theta\Lambda(\psi, \Gamma, \vartheta) \tag{7}$$

for all $\psi, \Gamma \in \Delta$ and $\vartheta > 0$.

Now, we prove the theorem for ENRC.

Theorem 2 *Let $(\Delta, \Omega, \Phi, \Lambda, *, \circ)$ be a complete ENRMS with $\wp: \Delta \times \Delta \to [1, +\infty)$ and suppose that*

$$\lim_{\vartheta \to +\infty} \Omega(\psi, \Gamma, \vartheta) = 1, \qquad \lim_{\vartheta \to +\infty} \Phi(\psi, \Gamma, \vartheta) = 0 \quad \text{and} \quad \lim_{\vartheta \to +\infty} \Lambda(\psi, \Gamma, \vartheta) = 0 \tag{8}$$

for all $\psi, \Gamma \in \Delta$ and $\vartheta > 0$. Let $\nabla: \Delta \to \Delta$ be an ENRC. Further, suppose that for an arbitrary $\psi_0 \in \Delta$ and $\kappa, \omega \in \mathbb{N}$, we have

$$\wp(\psi_\kappa, \psi_{\kappa+\omega}) < \frac{1}{\theta}.$$

Then ∇ has a unique fixed point.

Proof Let ψ_0 be a point of Δ and define a sequence ψ_κ by $\psi_\kappa = \nabla^\kappa \psi_0 = \nabla\psi_{\kappa-1}$, $\kappa \in \mathbb{N}$. By using (5), (6) and (7) for all $\vartheta > 0$, $\kappa > \omega$, we deduce

$$\begin{aligned}
&\frac{1}{\Omega(\psi_\kappa, \psi_{\kappa+1}, \vartheta)} - 1 \\
&\quad = \frac{1}{\Omega(\nabla\psi_{\kappa-1}, \nabla\psi_\kappa, \vartheta)} - 1 \le \theta\left[\frac{1}{\Omega(\psi_{\kappa-1}, \psi_\kappa, \vartheta)}\right] = \frac{\theta}{\Omega(\psi_{\kappa-1}, \psi_\kappa, \vartheta)} - \theta \\
&\quad \Rightarrow \quad \frac{1}{\Omega(\psi_\kappa, \psi_{\kappa+1}, \vartheta)} \\
&\qquad \le \frac{\theta}{\Omega(\psi_{\kappa-1}, \psi_\kappa, \vartheta)} + (1 - \theta) \le \frac{\theta^2}{\Omega(\psi_{\kappa-2}, \psi_{\kappa-1}, \vartheta)} + \theta(1 - \theta) + (1 - \theta).
\end{aligned}$$

Carrying on in this manner, we deduce

$$\begin{aligned}
\frac{1}{\Omega(\psi_\kappa, \psi_{\kappa+1}, \vartheta)} &\le \frac{\theta^\kappa}{\Omega(\psi_0, \psi_1, \vartheta)} + \theta^{\kappa-1}(1 - \theta) + \theta^{\kappa-2}(1 - \theta) + \cdots + \theta(1 - \theta) + (1 - \theta) \\
&\le \frac{\theta^\kappa}{\Omega(\psi_0, \psi_1, \vartheta)} + \left(\theta^{\kappa-1} + \theta^{\kappa-2} + \cdots + 1\right)(1 - \theta) \\
&\le \frac{\theta^\kappa}{\Omega(\psi_0, \psi_1, \vartheta)} + \left(1 - \theta^\kappa\right).
\end{aligned}$$

We obtain

$$\frac{1}{\frac{\theta^\kappa}{\Omega(\psi_0,\psi_1,\vartheta)} + (1-\theta^\kappa)} \leq \Omega(\psi_\kappa, \psi_{\kappa+1}, \vartheta), \tag{9}$$

$$\begin{aligned}\Phi(\psi_\kappa, \psi_{\kappa+1}, \vartheta) &= \Phi(\nabla\psi_{\kappa-1}, \nabla\psi_\kappa, \vartheta) \leq \theta\Phi(\psi_{\kappa-1}, \psi_\kappa, \vartheta) = \Phi(\nabla\psi_{\kappa-2}, \nabla\psi_{\kappa-1}, \vartheta) \\ &\leq \theta^2\Phi(\psi_{\kappa-2}, \psi_{\kappa-1}, \vartheta) \leq \cdots \leq \theta^\kappa\Phi(\psi_0, \psi_1, \vartheta) \end{aligned}\tag{10}$$

and

$$\begin{aligned}\Lambda(\psi_\kappa, \psi_{\kappa+1}, \vartheta) &= \Lambda(\nabla\psi_{\kappa-1}, \nabla\psi_\kappa, \vartheta) \leq \theta\Lambda(\psi_{\kappa-1}, \psi_\kappa, \vartheta) = \Lambda(\nabla\psi_{\kappa-2}, \nabla\psi_{\kappa-1}, \vartheta) \\ &\leq \theta^2\Lambda(\psi_{\kappa-2}, \psi_{\kappa-1}, \vartheta) \leq \cdots \leq \theta^\kappa\Lambda(\psi_0, \psi_1, \vartheta). \end{aligned}\tag{11}$$

Using (A5), (A10) and (A15), we have the following cases:

Case 1. When $\mathrm{i} = 2\pi + 1$, i.e. i is odd, then

$$\begin{aligned}&\Omega(\psi_\kappa, \psi_{\kappa+2\pi+1}, \vartheta) \\ &\quad\geq \Omega\left(\psi_\kappa, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\ &\qquad * \Omega\left(\psi_{\kappa+1}, \psi_{\kappa+2}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) * \cdots \\ &\qquad * \Omega\left(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi-1}, \frac{\vartheta}{3^\pi(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\ &\qquad * \Omega\left(\psi_{\kappa+2\pi-1}, \psi_{\kappa+2\pi}, \frac{\vartheta}{3^\pi(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\ &\qquad * \Omega\left(\psi_{\kappa+2\pi}, \psi_{\kappa+2\pi+1}, \frac{\vartheta}{3^\pi(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right), \\ &\Phi(\psi_\kappa, \psi_{\kappa+2\pi+1}, \vartheta) \\ &\quad\leq \Phi\left(\psi_\kappa, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \circ \Phi\left(\psi_{\kappa+1}, \psi_{\kappa+2}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\ &\qquad \circ \cdots \\ &\qquad \circ \Phi\left(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi-1}, \frac{\vartheta}{3^\pi(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\ &\qquad \circ \Phi\left(\psi_{\kappa+2\pi-1}, \psi_{\kappa+2\pi}, \frac{\vartheta}{3^\pi(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\ &\qquad \circ \Phi\left(\psi_{\kappa+2\pi}, \psi_{\kappa+2\pi+1}, \frac{\vartheta}{3^\pi(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \end{aligned}$$

and

$$\begin{aligned}&\Lambda(\psi_\kappa, \psi_{\kappa+2\pi+1}, \vartheta) \\ &\quad\leq \Lambda\left(\psi_\kappa, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \\ &\qquad \circ \Lambda\left(\psi_{\kappa+1}, \psi_{\kappa+2}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi+1}))}\right) \circ \cdots \end{aligned}$$

$$\circ \Lambda\left(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi-1}, \frac{\vartheta}{3^{\pi}(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right)$$
$$\circ \Lambda\left(\psi_{\kappa+2\pi-1}, \psi_{\kappa+2\pi}, \frac{\vartheta}{3^{\pi}(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right)$$
$$\circ \Lambda\left(\psi_{\kappa+2\pi}, \psi_{\kappa+2\pi+1}, \frac{\vartheta}{3^{\pi}(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right).$$

Using (4) in the above inequalities, we deduce

$$\begin{aligned}
&\Omega(\psi_{\kappa}, \psi_{\kappa+2\pi+1}, \vartheta) \\
&\geq \frac{1}{\frac{\theta^{\kappa}}{\Omega(\psi_0,\psi_1,\frac{\vartheta}{3(\wp(\psi_{\kappa},\psi_{\kappa+2\pi+1}))})} + (1-\theta^{\kappa})} * \frac{1}{\frac{\theta^{\kappa+1}}{\Omega(\psi_0,\psi_1,\frac{\vartheta}{3(\wp(\psi_{\kappa},\psi_{\kappa+2\pi+1}))})} + (1-\theta^{\kappa+1})} * \cdots \\
&\quad * \frac{1}{\frac{\theta^{\kappa+2\pi-2}}{\Omega(\psi_0,\psi_1,\frac{\vartheta}{3^{\pi}(\wp(\psi_{\kappa+2\pi-2},\psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4},\psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa},\psi_{\kappa+2\pi+1}))})} + (1-\theta^{\kappa+2\pi-2})} \\
&\quad * \frac{1}{\frac{\theta^{\kappa+2\pi-1}}{\Omega(\psi_0,\psi_1,\frac{\vartheta}{3^{\pi}(\wp(\psi_{\kappa+2\pi-2},\psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4},\psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa},\psi_{\kappa+2\pi+1}))})} + (1-\theta^{\kappa+2\pi-1})} \\
&\quad * \frac{1}{\frac{\theta^{\kappa+2\pi}}{\Omega(\psi_0,\psi_1,\frac{\vartheta}{3^{\pi}(\wp(\psi_{\kappa+2\pi-2},\psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4},\psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa},\psi_{\kappa+2\pi+1}))})} + (1-\theta^{\kappa+2\pi})},
\end{aligned}$$

$$\begin{aligned}
&\Phi(\psi_{\kappa}, \psi_{\kappa+2\pi+1}, \vartheta) \\
&\leq \theta^{\kappa}\Phi\left(\psi_0, \psi_1, \frac{\vartheta}{3(\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right) \\
&\quad \circ \theta^{\kappa+1}\Phi\left(\psi_0, \psi_1, \frac{\vartheta}{3(\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right) \circ \cdots \\
&\quad \circ \theta^{\kappa+2\pi-2}\Phi\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi}(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right) \\
&\quad \circ \theta^{\kappa+2\pi-1}\Phi\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi}(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right) \\
&\quad \circ \theta^{\kappa+2\pi}\Phi\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi}(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\Lambda(\psi_{\kappa}, \psi_{\kappa+2\pi+1}, \vartheta) \\
&\leq \theta^{\kappa}\Lambda\left(\psi_0, \psi_1, \frac{\vartheta}{3(\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right) \\
&\quad \circ \theta^{\kappa+1}\Lambda\left(\psi_0, \psi_1, \frac{\vartheta}{3(\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right) \circ \cdots \\
&\quad \circ \theta^{\kappa+2\pi-2}\Lambda\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi}(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right) \\
&\quad \circ \theta^{\kappa+2\pi-1}\Lambda\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi}(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right)
\end{aligned}$$

$$\circ\, \theta^{\kappa+2\pi} \Lambda\left(\psi_0, \psi_1, \frac{\vartheta}{3^{\pi}(\wp(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi+1})\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_{\kappa}, \psi_{\kappa+2\pi+1}))}\right).$$

Case 2. When $\mathfrak{i} = 2\pi$, i.e. $\mathfrak{i}$ is even, then

$$\begin{aligned}
&\Omega(\psi_\kappa, \psi_{\kappa+2\pi}, \vartheta) \\
&\quad \geq \Omega\left(\psi_\kappa, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) * \Omega\left(\psi_{\kappa+1}, \psi_{\kappa+2}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \\
&\qquad * \cdots \\
&\qquad * \Omega\left(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi-3}, \frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \\
&\qquad * \Omega\left(\psi_{\kappa+2\pi-3}, \psi_{\kappa+2\pi-2}, \frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \\
&\qquad * \Omega\left(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi}, \frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp\,(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right), \\
&\Phi(\psi_\kappa, \psi_{\kappa+2\pi}, \vartheta) \\
&\quad \leq \Phi\left(\psi_\kappa, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \circ \Phi\left(\psi_{\kappa+1}, \psi_{\kappa+2}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \\
&\qquad \circ \Phi\left(\psi_{\kappa+2}, \psi_{\kappa+3}, \frac{\vartheta}{3^2(\wp(\psi_{\kappa+2}, \psi_{\kappa+2\pi})\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \circ \cdots \\
&\qquad \circ \Phi\left(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi-3}, \frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \\
&\qquad \circ \Phi\left(\psi_{\kappa+2\pi-3}, \psi_{\kappa+2\pi-2}, \frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \\
&\qquad \circ \Phi\left(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi}, \frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\Lambda(\psi_\kappa, \psi_{\kappa+2\pi}, \vartheta) \\
&\quad \leq \Lambda\left(\psi_\kappa, \psi_{\kappa+1}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \circ \Lambda\left(\psi_{\kappa+1}, \psi_{\kappa+2}, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \\
&\qquad \circ \cdots \\
&\qquad \circ \Lambda\left(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi-3}, \frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \\
&\qquad \circ \Lambda\left(\psi_{\kappa+2\pi-3}, \psi_{\kappa+2\pi-2}, \frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right) \\
&\qquad \circ \Lambda\left(\psi_{\kappa+2\pi-2}, \psi_{\kappa+2\pi}, \frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4}, \psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6}, \psi_{\kappa+2\pi+1})\cdots\wp(\psi_\kappa, \psi_{\kappa+2\pi}))}\right).
\end{aligned}$$

Using (4) in the above inequalities, we deduce

$$\begin{aligned}
&\Omega(\psi_\kappa, \psi_{\kappa+2\pi}, \vartheta) \\
&\quad \geq \frac{1}{\frac{\theta^\kappa}{\Omega(\psi_0, \psi_1, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi}))})} + (1-\theta^\kappa)} * \frac{1}{\frac{\theta^{\kappa+1}}{\Omega(\psi_0, \psi_1, \frac{\vartheta}{3(\wp(\psi_\kappa, \psi_{\kappa+2\pi}))})} + (1-\theta^{\kappa+1})}
\end{aligned}$$

$$
\begin{aligned}
&* \frac{1}{\frac{\theta^{\kappa+2}}{\Omega(\psi_0,\psi_1,\frac{\vartheta}{3^2(\wp(\psi_{\kappa+2},\psi_{\kappa+2\pi})\wp(\psi_\kappa,\psi_{\kappa+2\pi}))})} + (1-\theta^{\kappa+2})} * \cdots * \\
&* \frac{1}{\frac{\theta^{\kappa+2\pi-4}}{\Omega(\psi_0,\psi_1,\frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4},\psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6},\psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa,\psi_{\kappa+2\pi}))})} + (1-\theta^{\kappa+2\pi-4})} \\
&* \frac{1}{\frac{\theta^{\kappa+2\pi-3}}{\Omega(\psi_0,\psi_1,\frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4},\psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6},\psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa,\psi_{\kappa+2\pi}))})} + (1-\theta^{\kappa+2\pi-3})} \\
&* \frac{1}{\frac{\theta^{\kappa+2\pi-2}}{\Omega(\psi_0,\psi_1,\frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4},\psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6},\psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa,\psi_{\kappa+2\pi}))})} + (1-\theta^{\kappa+2\pi-2})},
\end{aligned}
$$

$$
\begin{aligned}
&\Phi(\psi_\kappa,\psi_{\kappa+2\pi},\vartheta) \\
&\quad \le \theta^\kappa \Phi\left(\psi_0,\psi_1,\frac{\vartheta}{3(\wp(\psi_\kappa,\psi_{\kappa+2\pi}))}\right) \\
&\quad \circ \theta^{\kappa+1}\Phi\left(\psi_{\kappa+1},\psi_{\kappa+2},\frac{\vartheta}{3(\wp(\psi_\kappa,\psi_{\kappa+2\pi}))}\right) \circ \cdots \\
&\quad \circ \theta^{\kappa+2\pi-4}\Phi\left(\psi_0,\psi_1,\frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4},\psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6},\psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa,\psi_{\kappa+2\pi}))}\right) \\
&\quad \circ \theta^{\kappa+2\pi-3}\Phi\left(\psi_0,\psi_1,\frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4},\psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6},\psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa,\psi_{\kappa+2\pi}))}\right) \\
&\quad \circ \theta^{\kappa+2\pi-2}\Phi\left(\psi_0,\psi_1,\frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4},\psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6},\psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa,\psi_{\kappa+2\pi}))}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\Lambda(\psi_\kappa,\psi_{\kappa+2\pi},\vartheta) \\
&\quad \le \theta^\kappa \Lambda\left(\psi_0,\psi_1,\frac{\vartheta}{3(\wp(\psi_\kappa,\psi_{\kappa+2\pi}))}\right) \\
&\quad \circ \theta^{\kappa+1}\Lambda\left(\psi_{\kappa+1},\psi_{\kappa+2},\frac{\vartheta}{3(\wp(\psi_\kappa,\psi_{\kappa+2\pi}))}\right) \circ \cdots \\
&\quad \circ \theta^{\kappa+2\pi-4}\Lambda\left(\psi_0,\psi_1,\frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4},\psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6},\psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa,\psi_{\kappa+2\pi}))}\right) \\
&\quad \circ \theta^{\kappa+2\pi-3}\Lambda\left(\psi_0,\psi_1,\frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4},\psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6},\psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa,\psi_{\kappa+2\pi}))}\right) \\
&\quad \circ \theta^{\kappa+2\pi-2}\Lambda\left(\psi_0,\psi_1,\frac{\vartheta}{3^{\pi-1}(\wp(\psi_{\kappa+2\pi-4},\psi_{\kappa+2\pi})\wp(\psi_{\kappa+2\pi-6},\psi_{\kappa+2\pi})\cdots\wp(\psi_\kappa,\psi_{\kappa+2\pi}))}\right).
\end{aligned}
$$

Since $\kappa,\omega\in\mathbb{N}$, we have

$$\wp(\psi_\kappa,\psi_{\kappa+\omega}) < \frac{1}{\theta}.$$

Therefore, from (8), for each case $\kappa\to+\infty$, we deduce that

$$
\begin{aligned}
&\lim_{\kappa\to+\infty}\Omega(\psi_\kappa,\psi_{\kappa+\omega},\vartheta) = 1*1*\cdots* = 1, \\
&\lim_{\kappa\to+\infty}\Phi(\psi_\kappa,\psi_{\kappa+\omega},\vartheta) = 0\circ 0\circ\cdots\circ 0 = 0
\end{aligned}
$$

and

$$\lim_{\kappa\to+\infty} \Lambda(\psi_\kappa,\psi_{\kappa+\omega},\vartheta) = 0\circ 0\circ\cdots\circ 0 = 0.$$

Therefore, $\{\psi_\kappa\}$ is a Cauchy sequence. Since $(\Delta,\Omega,\Phi,\Lambda,*,\circ)$ is complete, there exists

$$\lim_{\kappa\to+\infty} \psi_\kappa = \psi.$$

From (A5), (A10) and (A15), we get

$$\begin{aligned}\frac{1}{\Omega(\nabla\psi_\kappa,\nabla\psi,\vartheta)} - 1 &\le \theta\left[\frac{1}{\Omega(\psi_\kappa,\psi,\vartheta)} - 1\right] = \frac{\theta}{\Omega(\psi_\kappa,\psi,\vartheta)} - \theta \\ &\Rightarrow \frac{1}{\frac{\theta}{\Omega(\psi_\kappa,\psi,\vartheta)} + (1-\theta)} \le \Omega(\nabla\psi_\kappa,\nabla\psi,\vartheta).\end{aligned}$$

Using the above inequality, we obtain

$$\begin{aligned}\Omega(\psi,\nabla\psi,\vartheta) &\ge \Omega\left(\psi,\psi_\kappa,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right) * \Omega\left(\psi_\kappa,\psi_{\kappa+1},\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right) \\ &\quad * \Omega\left(\psi_{\kappa+1},\nabla\psi,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right) \\ &\ge \Omega\left(\psi,\psi_\kappa,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right) * \Omega\left(\nabla\psi_{\kappa-1},\nabla\psi_\kappa,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right) \\ &\quad * \Omega\left(\nabla\psi_\kappa,\nabla\psi,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right) \\ &\ge \Omega\left(\psi,\psi_\kappa,\frac{\vartheta}{(3\wp(\psi,\nabla\psi)}\right) * \frac{1}{\frac{\theta^\kappa}{\Omega(\psi_0,\psi_1,\frac{\vartheta}{3\wp(\psi,\nabla\psi)})+(1-\theta^\kappa)}} \\ &\quad * \frac{1}{\frac{\theta}{\Omega(\psi_\kappa,\psi,\frac{\vartheta}{3\wp(\psi,\nabla\psi)})+(1-\theta)}} \\ &\to 1*1*1 = 1 \quad \text{as } \kappa\to+\infty, \\ \Phi(\psi,\nabla\psi,\vartheta) &\le \Phi\left(\psi,\psi_\kappa,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right) \circ \Phi\left(\psi_\kappa,\psi_{\kappa+1},\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right) \\ &\quad \circ \Phi\left(\psi_{\kappa+1},\nabla\psi,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right) \\ &\le \Phi\left(\psi,\psi_\kappa,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right) \circ \Phi\left(\nabla\psi_{\kappa-1},\nabla\psi_\kappa,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right) \\ &\quad \circ \Phi\left(\nabla\psi_\kappa,\nabla\psi,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right) \\ &\le \Phi\left(\psi,\psi_\kappa,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right) \circ \theta^{\kappa-1}\Phi\left(\psi_{\kappa-1},\psi_\kappa,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right) \\ &\quad \circ \theta\Phi\left(\psi_\kappa,\psi,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right) \\ &\to 0\circ 0\circ 0 = 0 \quad \text{as } \kappa\to+\infty\end{aligned}$$

and

$$\begin{aligned}
\Lambda(\psi,\nabla\psi,\vartheta) &\le \Lambda\left(\psi,\psi_\kappa,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right)\circ\Lambda\left(\psi_\kappa,\psi_{\kappa+1},\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right)\\
&\quad\circ\Lambda\left(\psi_{\kappa+1},\nabla\psi,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right)\\
&\le \Lambda\left(\psi,\psi_\kappa,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right)\circ\Lambda\left(\nabla\psi_{\kappa-1},\nabla\psi_\kappa,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right)\\
&\quad\circ\Lambda\left(\nabla\psi_\kappa,\nabla\psi,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right)\\
&\le \Lambda\left(\psi,\psi_\kappa,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right)\circ\theta^{\kappa-1}\Lambda\left(\psi_{\kappa-1},\psi_\kappa,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right)\\
&\quad\circ\theta\Lambda\left(\psi_\kappa,\psi,\frac{\vartheta}{3\wp(\psi,\nabla\psi)}\right)\\
&\to 0\circ 0\circ 0=0\quad\text{as }\kappa\to+\infty.
\end{aligned}$$

Hence, $\nabla\psi=\psi$. Let $\nabla\mu=\mu$ for some $\mu\in\Delta$, then

$$\begin{aligned}
\frac{1}{\Omega(\psi,\mu,\vartheta)}-1 &= \frac{1}{\Omega(\nabla\psi,\nabla\mu,\vartheta)}-1\\
&\le \theta\left[\frac{1}{\Omega(\psi,\mu,\vartheta)}-1\right] < \frac{1}{\Omega(\psi,\mu,\vartheta)}-1,
\end{aligned}$$

which is a contradiction.

$$\Phi(\psi,\mu,\vartheta)=\Phi(\nabla\psi,\nabla\mu,\vartheta)\le\theta\Phi(\psi,\mu,\vartheta)<\Phi(\psi,\mu,\vartheta),$$

which is a contradiction and

$$\Lambda(\psi,\mu,\vartheta)=\Lambda(\nabla\psi,\nabla\mu,\vartheta)\le\theta\Lambda(\psi,\mu,\vartheta)<\Lambda(\psi,\mu,\vartheta),$$

which is a contradiction. Therefore, $\Omega(\psi,\mu,\vartheta)=1$, $\Phi(\psi,\mu,\vartheta)=0$ and $\Lambda(\psi,\mu,\vartheta)=0$, hence, $\psi=\mu$. Hence, ∇ has a unique fixed point. □

Example 2 Let $\Delta=[0,1]$ and $\wp:\Delta\times\Delta\to[1,+\infty)$ be a function given by

$$\wp(\psi,\Gamma)=\begin{cases}1 & \text{if }\psi=\Gamma,\\ \frac{1+\max\{\psi,\Gamma\}}{1+\min\{\psi,\Gamma\}} & \text{if }\psi\ne\Gamma.\end{cases}$$

Define $\Omega,\Phi,\Lambda:\Delta\times\Delta\times(0,+\infty)\to[0,1]$ as

$$\begin{aligned}
\Omega(\psi,\Gamma,\vartheta)&=\frac{\vartheta}{\vartheta+|\psi-\Gamma|^2},\\
\Phi(\psi,\Gamma,\vartheta)&=\frac{|\psi-\Gamma|}{\vartheta+|\psi-\Gamma|^2},\\
\Lambda(\psi,\Gamma,\vartheta)&=\frac{|\psi-\Gamma|^2}{\vartheta}.
\end{aligned}$$

Then $(\Delta,\Omega,\Phi,\Lambda,*,\circ)$ is a complete ENRMS with continuous t-norm $\wp*\tau=\wp\tau$ and continuous t-co-norm $\wp\circ\tau=\max\{\wp,\tau\}$.

Define $\nabla\colon\Delta\to\Delta$ by $\nabla(\psi)=\frac{1-3^{-\psi}}{7}$ and take $\theta\in[\frac{1}{2},1)$, then

$$\begin{aligned}
\Omega(\nabla\psi,\nabla\Gamma,\theta\vartheta)&=\Omega\left(\frac{1-3^{-\psi}}{7},\frac{1-3^{-\Gamma}}{7},\theta\vartheta\right)\\
&=\frac{\theta\vartheta}{\theta\vartheta+|\frac{1-3^{-\psi}}{7}-\frac{1-3^{-\Gamma}}{7}|^2}=\frac{\theta\vartheta}{\theta\vartheta+\frac{|3^{-\psi}-3^{-\Gamma}|^2}{49}}\\
&\geq\frac{\theta\vartheta}{\theta\vartheta+\frac{|\psi-\Gamma|^2}{49}}=\frac{49\theta\vartheta}{49\theta\vartheta+|\psi-\Gamma|}\geq\frac{\vartheta}{\vartheta+|\psi-\Gamma|}=\Omega(\psi,\Gamma,\vartheta),\\
\Phi(\nabla\psi,\nabla\Gamma,\theta\vartheta)&=\Phi\left(\frac{1-3^{-\psi}}{7},\frac{1-3^{-\Gamma}}{7},\theta\vartheta\right)\\
&=\frac{|\frac{1-3^{-\psi}}{7}-\frac{1-3^{-\Gamma}}{7}|^2}{\theta\vartheta+|\frac{1-3^{-\psi}}{7}-\frac{1-3^{-\Gamma}}{7}|^2}=\frac{\frac{|3^{-\psi}-3^{-\Gamma}|^2}{49}}{\theta\vartheta+\frac{|3^{-\psi}-3^{-\Gamma}|^2}{49}}\\
&=\frac{|3^{-\psi}-3^{-\Gamma}|^2}{49\theta\vartheta+|3^{-\psi}-3^{-\Gamma}|^2}\\
&\leq\frac{|\psi-\Gamma|^2}{49\theta\vartheta+|\psi-\Gamma|^2}\leq\frac{|\psi-\Gamma|^2}{\vartheta+|\psi-\Gamma|^2}=\Phi(\psi,\Gamma,\vartheta)
\end{aligned}$$

and

$$\begin{aligned}
\Lambda(\nabla\psi,\nabla\Gamma,\theta\vartheta)&=\Lambda\left(\frac{1-3^{-\psi}}{7},\frac{1-3^{-\Gamma}}{7},\theta\vartheta\right)\\
&=\frac{|\frac{1-3^{-\psi}}{7}-\frac{1-3^{-\Gamma}}{7}|^2}{\theta\vartheta}=\frac{\frac{|3^{-\psi}-3^{-\Gamma}|^2}{49}}{\theta\vartheta}\\
&=\frac{|3^{-\psi}-3^{-\Gamma}|^2}{49\theta\vartheta}\leq\frac{|\psi-\Gamma|}{49\theta\vartheta}\leq\frac{|\psi-\Gamma|}{\vartheta}=\Lambda(\psi,\Gamma,\vartheta).
\end{aligned}$$

As a result, all of the conditions of Theorem 1 are satisfied, and 0 is the only fixed point for ∇.

4 Application to fractional differential equations

This section is devoted to finding a solution of the following fractional differential equation consisting of Caputo fractional derivative. Further details can be found in [29].

$$\mathcal{D}^{\delta}_{0+}\psi(\tau)+\mathfrak{g}\big(\tau,\psi(\tau)\big)=0,\quad 0<\tau<1,\tag{12}$$

where $1<\delta\leq 2$, $\psi(0)+\psi'(0)=0$, $\psi(1)+\psi'(1)=0$ are the boundary conditions with $\mathfrak{g}\colon[0,1]\times[0,\infty)\to[0,\infty)$ being continuous. Define Ω, Φ and Λ given by

$$\Omega\big(\psi(\tau),\Gamma(\tau),\vartheta\big)=\sup_{\tau\in[\mathfrak{c},\mathfrak{a}]}\frac{\vartheta}{\vartheta+|\psi(\tau)-\Gamma(\tau)|^2}\quad\text{for all }\psi,\Gamma\in\Delta\text{ and }\vartheta>0,$$

$$\Phi\big(\psi(\tau),\Gamma(\tau),\vartheta\big)=1-\sup_{\tau\in[\mathfrak{c},\mathfrak{a}]}\frac{\vartheta}{\vartheta+|\psi(\tau)-\Gamma(\tau)|^2}\quad\text{for all }\psi,\Gamma\in\Delta\text{ and }\vartheta>0$$

and

$$\Lambda\big(\psi(\tau),\Gamma(\tau),\vartheta\big)=\sup_{\tau\in[\mathfrak{c},\mathfrak{a}]}\frac{|\psi(\tau)-\Gamma(\tau)|^2}{\vartheta}\quad\text{for all }\psi,\Gamma\in\Delta\text{ and }\vartheta>0,$$

with continuous t-norm and continuous t-co-norm defined by $\hat{e}*\breve{a}=\hat{e}\cdot\breve{a}$ and $\hat{e}\circ\breve{a}=\max\{\hat{e},\breve{a}\}$, respectively. Define $\wp\colon\Delta\times\Delta\to[1,+\infty)$ as

$$\wp(\psi,\Gamma)=\psi+\Gamma+1.$$

Then $(\Delta,\Omega,\Phi,\Lambda,*,\circ)$ is a complete ENRMS. Note that $\psi\in\Delta$ solves (12) whenever $\psi\in\Delta$ is the solution of

$$\begin{aligned}\psi(\tau)&=\frac{1}{\Gamma(\delta)}\int_0^1(1-\zeta)^{\delta-1}(1-\tau)\mathfrak{g}\big(\zeta,\psi(\zeta)\big)\,d\zeta\\&\quad+\frac{1}{\Gamma(\delta-1)}\int_0^1(1-\zeta)^{\delta-2}(1-\tau)\mathfrak{g}\big(\zeta,\psi(\zeta)\big)\,d\zeta\\&\quad+\frac{1}{\Gamma(\delta)}\int_0^\tau(\tau-\zeta)^{\delta-1}\mathfrak{g}\big(\zeta,\psi(\zeta)\big)\,d\zeta.\end{aligned}$$

Theorem 3 *Consider the operator* $\nabla\colon\Delta\to\Delta$ *as*

$$\begin{aligned}\nabla\psi(\tau)&=\frac{1}{\Gamma(\delta)}\int_0^1(1-\zeta)^{\delta-1}(1-\tau)\mathfrak{g}\big(\zeta,\psi(\zeta)\big)\,d\zeta\\&\quad+\frac{1}{\Gamma(\delta-1)}\int_0^1(1-\zeta)^{\delta-2}(1-\tau)\mathfrak{g}\big(\zeta,\psi(\zeta)\big)\,d\zeta\\&\quad+\frac{1}{\Gamma(\delta)}\int_0^\tau(\tau-\zeta)^{\delta-1}\mathfrak{g}\big(\zeta,\psi(\zeta)\big)\,d\zeta.\end{aligned}$$

Suppose that the conditions:

(i) *for all* $\psi,\Gamma\in\Delta$, $\mathfrak{g}\colon[0,1]\times[0,\infty)\to[0,\infty)$ *and* $\theta\in(0,1)$ *satisfies*

$$\big|\mathfrak{g}\big(\zeta,\psi(\zeta)\big)-\mathfrak{g}\big(\zeta,\Gamma(\zeta)\big)\big|\le\sqrt{\theta}\big|\psi(\zeta)-\Gamma(\zeta)\big|;$$

(ii)

$$\sup_{\tau\in(0,1)}\left|\frac{1-\tau}{\Gamma(\delta+1)}+\frac{1-\tau}{\Gamma(\delta)}+\frac{\tau^\delta}{\Gamma(\delta+1)}\right|^2=\eta<1,$$

hold. Then equation (1) *has a unique solution.*

Proof Let $\psi,\Gamma\in\Delta$ and consider

$$\begin{aligned}\big|\nabla\psi(\tau)-\nabla\Gamma(\tau)\big|^2&=\bigg|\frac{1}{\Gamma(\delta)}\int_0^1(1-\zeta)^{\delta-1}(1-\tau)\big(\mathfrak{g}\big(\zeta,\psi(\zeta)\big)-\mathfrak{g}\big(\zeta,\Gamma(\zeta)\big)\big)\,d\zeta\\&\quad+\frac{1}{\Gamma(\delta-1)}\int_0^1(1-\zeta)^{\delta-2}(1-\tau)\big(\mathfrak{g}\big(\zeta,\psi(\zeta)\big)-\mathfrak{g}\big(\zeta,\Gamma(\zeta)\big)\big)\,d\zeta\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{1}{\Gamma(\delta)}\int_0^\tau (\tau-\zeta)^{\delta-1}\big(\mathfrak{g}\big(\zeta,\psi(\zeta)\big)-\mathfrak{g}\big(\zeta,\Gamma(\zeta)\big)\big)\,d\zeta\bigg|^2 \\
&\leq \bigg(\frac{1}{\Gamma(\delta)}\int_0^1 (1-\zeta)^{\delta-1}(1-\tau)\big|\big(\mathfrak{g}\big(\zeta,\psi(\zeta)\big)-\mathfrak{g}\big(\zeta,\Gamma(\zeta)\big)\big)\big|\,d\zeta \\
&\quad + \frac{1}{\Gamma(\delta-1)}\int_0^1 (1-\zeta)^{\delta-2}(1-\tau)\big|\big(\mathfrak{g}\big(\zeta,\psi(\zeta)\big)-\mathfrak{g}\big(\zeta,\Gamma(\zeta)\big)\big)\big|\,d\zeta \\
&\quad + \frac{1}{\Gamma(\delta)}\int_0^\tau (\tau-\zeta)^{\delta-1}\big|\big(\mathfrak{g}\big(\zeta,\psi(\zeta)\big)-\mathfrak{g}\big(\zeta,\Gamma(\zeta)\big)\big)\big|\,d\zeta\bigg)^2 \\
&\leq \bigg(\frac{1}{\Gamma(\delta)}\int_0^1 (1-\zeta)^{\delta-1}(1-\tau)\theta^{\frac{1}{2}}\big|\psi(\zeta)-\Gamma(\zeta)\big|\,d\zeta \\
&\quad + \frac{1}{\Gamma(\delta-1)}\int_0^1 (1-\zeta)^{\delta-2}(1-\tau)\theta^{\frac{1}{2}}\big|\psi(\zeta)-\Gamma(\zeta)\big|\,d\zeta \\
&\quad + \frac{1}{\Gamma(\delta)}\int_0^\tau (\tau-\zeta)^{\delta-1}\theta^{\frac{1}{2}}\big|\psi(\zeta)-\Gamma(\zeta)\big|\,d\zeta\bigg)^2 \\
&= \theta\big|\psi(\tau)-\Gamma(\tau)\big|^2\bigg(\frac{1}{\Gamma(\delta)}\int_0^1 (1-\zeta)^{\delta-1}(1-\tau)\,d\zeta \\
&\quad + \frac{1}{\Gamma(\delta-1)}\int_0^1 (1-\zeta)^{\delta-2}(1-\tau)\,d\zeta + \frac{1}{\Gamma(\delta)}\int_0^\tau (\tau-\zeta)^{\delta-1}\,d\zeta\bigg)^2 \\
&= \theta\big|\psi(\tau)-\Gamma(\tau)\big|^2\bigg(\frac{1-\tau}{\Gamma(\delta+1)}+\frac{1-\tau}{\Gamma(\delta)}+\frac{\tau^\delta}{\Gamma(\delta+1)}\bigg)^2 \\
&\leq \theta\big|\psi(\tau)-\Gamma(\tau)\big|^2 \sup_{\tau\in(0,1)}\bigg(\frac{1-\tau}{\Gamma(\delta+1)}+\frac{1-\tau}{\Gamma(\delta)}+\frac{\tau^\delta}{\Gamma(\delta+1)}\bigg)^2 \\
&= \eta\theta\big|\psi(\tau)-\Gamma(\tau)\big|^2 \\
&\leq \theta\big|\psi(\tau)-\Gamma(\tau)\big|^2.
\end{aligned}
$$

So, we have

$$
\big|\nabla\psi(\tau)-\nabla\Gamma(\tau)\big|^2 \leq \theta\big|\psi(\tau)-\Gamma(\tau)\big|^2.
$$

Now, for all $\psi, \Gamma \in \Delta$, we deduce

$$
\begin{aligned}
\Omega\big(\nabla\psi(\tau),\nabla\Gamma(\tau),\theta\vartheta\big) &= \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]}\frac{\theta\vartheta}{\theta\vartheta+|\nabla\psi(\tau)-\nabla\Gamma(\tau)|^2} \\
&\geq \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]}\frac{\theta\vartheta}{\theta\vartheta+\theta|\psi(\tau)-\Gamma(\tau)|^2} \\
&= \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]}\frac{\vartheta}{\vartheta+|\psi(\tau)-\Gamma(\tau)|^2} \\
&= \Omega\big(\psi(\tau),\Gamma(\tau),\vartheta\big), \\
\Phi\big(\nabla\psi(\tau),\nabla\Gamma(\tau),\theta\vartheta\big) &= 1-\sup_{\tau\in[\mathfrak{c},\mathfrak{a}]}\frac{\theta\vartheta}{\theta\vartheta+|\nabla\psi(\tau)-\nabla\Gamma(\tau)|^2} \\
&\leq 1-\sup_{\tau\in[\mathfrak{c},\mathfrak{a}]}\frac{\theta\vartheta}{\theta\vartheta+\theta|\psi(\tau)-\Gamma(\tau)|^2}
\end{aligned}
$$

$$\begin{aligned} &= 1 - \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\vartheta}{\vartheta + |\psi(\tau) - \Gamma(\tau)|^2} \\ &= \Phi\bigl(\psi(\tau), \Gamma(\tau), \vartheta\bigr) \end{aligned}$$

and

$$\begin{aligned} \Lambda\bigl(\nabla\psi(\tau), \nabla\Gamma(\tau), \theta\vartheta\bigr) &= \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\theta\vartheta}{\theta\hat{r} + |\nabla\psi(\tau) - \nabla\Gamma(\tau)|^2} \\ &\leq \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\theta\hat{r}}{\theta\hat{r} + \theta|\psi(\tau) - \Gamma(\tau)|^2} \\ &= \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\hat{r}}{\hat{r} + |\psi(\tau) - \Gamma(\tau)|^2} \\ &= \Omega\bigl(\psi(\tau), \Gamma(\tau), \hat{r}\bigr). \end{aligned}$$

As a result, all of the conditions of Theorem 1 are satisfied and operator ∇ has a unique fixed point. □

Example 3 According to equation (12), we consider

$$\begin{cases} \mathcal{D}^{\delta}_{0+}\psi(\tau) + \frac{\sqrt{6}\ln(\tau+1)\cos(\psi(\tau))}{\tau^2+1} = 0, & 0 \leq \tau \leq 1, \\ \psi(0) + \psi'(0) = 0, \qquad \psi(1) + \psi'(1) = 0, \end{cases} \tag{13}$$

with three cases $\delta = \{\frac{25}{19}, \frac{7}{4}, \frac{35}{19}\}$, where $\mathfrak{g}(\tau, \psi(\tau)) = \frac{\sqrt{6}\ln(\tau+1)\cos(\psi(\tau))}{\tau^2+1}$. Then, for ψ, $\Gamma \in \Delta = [0,1]$, we have

$$\begin{aligned} \bigl|\mathfrak{g}\bigl(\tau, \psi(\tau)\bigr) - \mathfrak{g}\bigl(\tau, \Gamma(\tau)\bigr)\bigr| &= \left| \frac{\sqrt{6}\ln(\tau+1)\cos(\psi(\tau))}{\tau^2+1} \right. \\ &\qquad \left. - \frac{\sqrt{6}\ln(\tau+1)\cos(\Gamma(\tau))}{\tau^2+1} \right| \\ &= \left| \frac{\sqrt{6}\ln(\tau+1)}{\tau^2+1} \right| \bigl|\cos\bigl(\psi(\tau)\bigr) - \cos\bigl(\Gamma(\tau)\bigr)\bigr| \\ &\leq \sqrt{6}\ln 2\bigl|\psi(\tau) - \Gamma(\tau)\bigr| = \sqrt{\theta}\bigl|\psi(\tau) - \Gamma(\tau)\bigr|, \end{aligned}$$

where $\theta = (\sqrt{6}\ln 2)^2$. Therefore, all the conditions of Theorem 3 are true. Hence, ∇ has a unique fixed point.

5 Conclusion

This paper introduced the concept of ENRMS, as well as various new types of fixed point theorems that can be proved in this novel environment. Furthermore, we offered a non-trivial example to show that the proposed solutions are viable. We have complemented our work with an application that shows how the developed approach outperforms the literature-based methods. It is an interesting open problem to prove a coupled fixed point under this space.

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Author contributions
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